Math 522 Exam 3 Solutions

1. Calculate and simplify $\binom{1/3}{4}$.

   $$\binom{1/3}{4} = \frac{(1/3)(-2/3)(-5/3)(-8/3)}{4!} = \frac{-80}{3^4 \cdot 24} = -\frac{80}{1944} = -\frac{10}{243}.$$

2. Let $NP = \mathbb{N} \setminus \mathbb{P}$ denote the set of non-prime natural numbers, under the multiplication operation. $NP = \{1, 4, 6, 8, 9, 10, \ldots\}$.

   (a) For any (not necessarily distinct) primes $p, q \in \mathbb{P}$, prove that pq is irreducible in NP.

   (b) For any (not necessarily distinct) primes $p, q, r \in \mathbb{P}$, prove that pqr is irreducible in NP.

   (c) Find a factorization of $240 = 2^4 \cdot 3 \cdot 5$ into two irreducibles, and another factorization into three irreducibles.

   (d) BONUS: 240 has two different factorization lengths; find a number with three different factorization lengths.

   The two proofs rely on the fact that $NP \subseteq \mathbb{N}$; if $x = yz$ is reducible in NP, then x must be reducible in $\mathbb{N}$ (but not the converse!). Hence pq can only be reduced as $p \cdot q$ in $\mathbb{N}$, but neither of these factors is in NP. Similarly, pqr can be reduced in only three ways in $\mathbb{N}$: $(pq) \cdot r$ or $(pr) \cdot q$ or $(qr) \cdot p$. Each of these contains a natural number that is not in NP.

   The factorizations can be done in many ways, relying on the first two parts of the problem. There are six primes in 240, so they must be grouped into two groups of three, or three groups of two. For example, $8 \cdot 30 = (2^3) \cdot (2 \cdot 3 \cdot 5)$ is a factorization into two irreducibles, and $4 \cdot 6 \cdot 10 = (2^2) \cdot (2 \cdot 3) \cdot (2 \cdot 5)$ is a factorization into three irreducibles.

   This object, NP, is not a ring (lacking addition), or a group (lacking division), but it is something called a semigroup.

   For the bonus, you need a number with at least twelve prime divisors. They can be grouped into six pairs (six irreducibles), or three pairs and two triples (five irreducibles), or four triples (four irreducibles). The smallest example is $4096 = 2^{12} = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4 \cdot 4 \cdot 4 \cdot 8 \cdot 8 = 8 \cdot 8 \cdot 8 \cdot 8$.

3. High score=99, Median score=85, Low score=50